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Structural Reanalysis for General Layout Modifications

Uri Kirsch* and Shenghu Liu†
Technion—Israel Institute of Technology, Haifa 32000, Israel

A procedure for reanalysis of structures subjected to various layout modifications is presented. The procedure is based on exact analysis of the initial design. It is suitable for all types of layout changes, including the general case in which some members and joints are added, other members and joints are deleted, and some joint coordinates are modified. In the particular and challenging case of addition of joints, where the number of degrees of freedom is increased, a simple reanalysis of a modified initial design is first carried out. Further changes are then applied to the latter design. It is shown that, for changes in a limited number of members of a truss structure, an exact solution is readily achieved. For changes in numerous members of a general structure, an approximate solution can be efficiently obtained. High-quality approximations are demonstrated for various types of layout modifications.

Introduction

Layout Optimization and Reanalysis

LAYOUT optimization of structures has been the subject of numerous studies in recent years.[1] It is recognized that this type of optimization can greatly improve the design, and potential savings are generally more significant than those resulting from fixed-layout optimization. However, the solution of layout optimization problems is more difficult because of changes in the structural model. In particular, changes in the numbers of variables and degrees of freedom (DOF) result in corresponding changes in the form of the analysis equations and constraint functions.

One of the main obstacles in structural optimization, particularly in layout optimization, is the high computational cost involved in repeated analyses required during the solution of large-scale problems. In many problems, it is necessary to analyze repeatedly modified structures because of various types of changes, including cross sections of elements, the geometry of the structure (coordinates of joints), or topology (number and orientation of elements).

Reanalysis methods are intended to analyze efficiently modified designs. The object is to evaluate the structural response (e.g., displacements, stresses, and forces) for successive modifications in the design without solving the set of modified implicit equations. Approximate reanalysis methods, intended to reduce the computational cost, have been motivated by the following characteristics of structural optimization problems.

1) The problem size (number of variables and constraints) is usually large.
2) Each redesign involves extensive calculations.
3) The number of redesigns is large.

In layout optimization, members and joints are added or deleted during the solution process, and the geometry of the structure is modified because of changes in joint coordinates. Developing a reanalysis procedure for general layout modifications is most challenging, particularly in cases in which the number of DOF is modified and the structural behavior is significantly changed. For such modifications, approximate reanalysis methods[2,3] usually are not suitable, because they provide inadequate or even meaningless results.

Considering a general layout optimization problem, the various modifications in the structure can be classified as follows.

1) Deletion of members and joints, where both the design variable vector and the number of DOF are reduced. If only members are deleted, the value of some design variables becomes zero and can be eliminated from the set of variables, but the analysis model is unchanged.

2) Addition of members and joints, where both the design variable vector and the number of DOF are increased. If only members are added, then the vector of design variables is expanded, but the number of DOF is unchanged.

3) Modification in the geometry, where there is no change in the number of variables and in the number of DOF. In this case, only the numerical values of the variables are modified.

In previous studies,[3–9] reanalysis procedures have been developed for the following cases of layout modifications:

a) Approximate and exact solutions for the case of deletion of members and joints (case 1).

b) Approximate and exact solutions for the case of addition of members only (case 2).

c) Approximate solution for modifications in the geometry of the structure (case 3).

In a general layout optimization problem, it might be necessary to consider all types of changes simultaneously. That is, some members and joints are added, other members and joints are deleted, and some joint coordinates are modified. In this paper, a reanalysis procedure for such modifications is presented. In particular, reanalysis models for the following cases that have not been solved in previous studies are developed:

a) Approximate and exact solutions for the case of addition of members and joints (case 2).

b) Exact solution for modifications in the geometry of the structure (case 3).

c) A general solution procedure, including all types of changes.

An exact solution is readily achieved for changes in a limited number of members of a truss structure. For changes in numerous members of a general structure, an approximate solution can efficiently be obtained. High-quality approximations are demonstrated for various types of layout modifications.

Reanalysis Methods

Reanalysis methods can be broadly divided into approximate and exact (direct-closed form) methods. Approximate methods are usually suitable for moderate changes in numerous design variables, whereas exact methods are efficient only for changes in a relatively small proportion of the structure. In this section, some available methods are briefly described. Comprehensive reviews are given elsewhere.[2,10–12]

Several exact methods for calculating the modified behavior as a result of changes in the structure have been proposed in the past. Most of these methods are based on the Sherman-Morrison identity[13] and are suitable for cases in which the changes can be represented by a small submatrix of changes in stiffness. Improved versions of this approach have been proposed by several authors.[14–17] One limitation of these methods is that they are inefficient when the submatrix of changes is large. Force expressions for a change in a single member of a truss, based on the force method of analysis, have been proposed by Fuchs and Steinberg.[18]

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*Professor, Department of Civil Engineering.

†Formerly Graduate Student, Department of Civil Engineering.

Approximate reanalysis methods are usually more efficient than exact methods. Considering the former methods, the quality of the results and the efficiency of the calculations are usually two conflicting factors. That is, better approximations are often achieved at the expense of more computational effort. The various approximations can be divided into the following classes[2,3]:

1) Global approximations (called also multipoint approximations), such as polynomial fitting or reduced basis methods.[19–21] These approximations are obtained by analyzing the structure at a number of design points, and they are valid for the whole design space (or, at least, large regions of it). However, global approximations may require much computational effort in problems with a large number of design modifications.

2) Local approximations (called also single-point approximations), such as the first-order Taylor series expansion or the binomial series expansion about a given design point. Local approximations are based on information calculated at a single point. These methods are most efficient, but they are effective only in cases of small changes in design variables. For large changes in design, the accuracy of the approximations often deteriorates and they may become meaningless. That is, the approximations are valid only in the vicinity of a design point. To improve the quality of the results, reciprocal cross-sectional areas are often assumed as design variables.[22,23] A hybrid form of the direct and reciprocal approximations, which is more conservative than either, also can be introduced.[24] This approximation has the advantage of being convex,[25] but it has been found that the hybrid approximation tends to be less accurate than either the direct or the reciprocal approximation. More accurate convex approximations can be introduced by the method of moving asymptotes,[26] but the quality of the results is highly dependent on selection of the moving asymptotes. Another possibility to improve the quality of the results is to consider second-order approximations,[27] but this might increase considerably the computational effort.

3) In this paper, a third class, called combined approximations (CA), attempting to give global qualities to local approximations, is presented. The method, used in previous studies for structural optimization,[4–9] is suitable for different types of design variables such as cross-sectional changes as well as geometrical and topological modifications. Similar to local approximations, the calculations are based on results of a single exact analysis. Each subsequent reanalysis involves the solution of only a small system of equations. Thus, the computational effort is significantly reduced. Calculation of derivatives is not required, and the method can be used with a general finite element program. Recently, it has been found that the method provides the exact solution in certain cases.[9]

Problem Formulation

Assume an initial design variables vector $\boldsymbol{X}_0$, the corresponding stiffness matrix $\boldsymbol{K}_0$, and the displacement vector $\boldsymbol{r}_0$, computed by the equilibrium equations

$$\boldsymbol{K}_0\boldsymbol{r}_0 = \boldsymbol{R} \tag{1}$$

The elements of the load vector $\boldsymbol{R}$ are assumed to be independent of the design variables. However, the approach presented here is suitable also for cases of changes in the load vector. The stiffness matrix $\boldsymbol{K}_0$ is often given from the initial analysis in the decomposed form

$$\boldsymbol{K}_0 = \boldsymbol{U}_0^T\boldsymbol{U}_0 \tag{2}$$

where $\boldsymbol{U}_0$ is an upper triangular matrix.

Most reanalysis methods developed in the past are suitable for the relatively simple case where the structural model (or the number of DOF) and the size of $\boldsymbol{K}_0$, $\boldsymbol{r}_0$, and $\boldsymbol{R}$ are unchanged. The reanalysis method developed in this study is intended for problems where the size of the above quantities is changed as a result of changes in the number of DOF. In the formulation presented in this section, a distinction is made between the following two cases:

1) The common case where the number of DOF is not increased.

2) The more challenging case considered in this paper, where the number of DOF is increased.

Number of DOF Is Not Increased

Assume a change $\Delta\boldsymbol{X}$ in the design variables so that the modified design is

$$\boldsymbol{X} = \boldsymbol{X}_0 + \Delta\boldsymbol{X} \tag{3}$$

and the corresponding stiffness matrix is

$$\boldsymbol{K} = \boldsymbol{K}_0 + \Delta\boldsymbol{K} \tag{4}$$

where $\Delta\boldsymbol{K}$ is the change in the stiffness matrix due to the change $\Delta\boldsymbol{X}$.

The problem under consideration can be formulated as follows. Given $\boldsymbol{K}_0$ and $\boldsymbol{r}_0$, the object is to find efficient and high-quality approximations of the modified displacements $\boldsymbol{r}$ that result from various changes in the design variables $\Delta\boldsymbol{X}$ without solving the modified analysis equations

$$\boldsymbol{K}\boldsymbol{r} = (\boldsymbol{K}_0 + \Delta\boldsymbol{K})\boldsymbol{r} = \boldsymbol{R} \tag{5}$$

The elements of the stiffness matrix are not restricted to certain forms and can be general functions of the design variables; that is, the design variables $\boldsymbol{X}$ may represent coordinates of joints, the structural shape, members' cross sections, etc.

The reanalysis model presented here is intended to replace the implicit analysis [Eq. (5)]. Once the displacements $\boldsymbol{r}$ are evaluated, the stresses σ can be calculated by the explicit stress-displacement relations

$$\sigma = \boldsymbol{S}\boldsymbol{r} \tag{6}$$

in which $\boldsymbol{S}$ is the stress-transformation matrix. The elements of matrices $\boldsymbol{K}$ and $\boldsymbol{S}$ are some explicit functions of the design variables. Because the stresses are explicit functions of $\boldsymbol{S}$ and $\boldsymbol{r}$, they can readily be evaluated.

Number of DOF Is Increased

Consider the case where a new joint and some members connecting this joint to existing joints are added to the initial structure so that the number of DOF is increased. The modified analysis equations are

$$\boldsymbol{K}_M\boldsymbol{r}_M = \boldsymbol{R}_M \tag{7}$$

in which subscript M denotes quantities related to the modified design. If the new joint is not loaded, then the modified load vector can be expressed as

$$\boldsymbol{R}_M = \begin{Bmatrix} \boldsymbol{R} \\ \boldsymbol{0} \end{Bmatrix} \tag{8}$$

In addition, the modified stiffness matrix can be partitioned to obtain

$$\boldsymbol{K}_M = \begin{bmatrix} \boldsymbol{K}_{00} & \boldsymbol{K}_{0M} \\ \boldsymbol{K}_{M0} & \boldsymbol{K}_{MM} \end{bmatrix} \tag{9}$$

where $\boldsymbol{K}_{00} = \boldsymbol{K}_0$ is a submatrix of the original DOF and $\boldsymbol{K}_{MM}$ is a submatrix of stiffness coefficients of the new joint.

The object is to find the modified displacements $\boldsymbol{r}_M$ that result from addition of the new joint and members without solving the modified analysis [Eq. (7)]. Evidently, developing a reanalysis method in this case is more challenging because both the size and the numerical values of the elements of the displacement vector are changed.

In the next section, the approach proposed to solve this problem is presented. For completeness of presentation, the approach developed in previous studies for the common case, where the number of DOF is not increased, is first described. Then the method developed in this study, for the case where the number of DOF is increased, is presented.

Solution Approach

Number of DOF Is Not Increased

Approximate Solution

The CA approach has previously been demonstrated[3–5] for problems with unchanged numbers of design variables and DOF. The solution procedure is based on combining the reduced basis method[19,21] and the first terms of a series expansion. Assuming the reduced basis method and considering second-order approximations, the displacements are expressed as

$$\boldsymbol{r} = y_0\boldsymbol{r}_0 + y_1\boldsymbol{r}_1 + y_2\boldsymbol{r}_2 = \boldsymbol{r}_B\boldsymbol{y} \tag{10}$$

where $\boldsymbol{r}_0$, $\boldsymbol{r}_1$, and $\boldsymbol{r}_2$ are the first three terms of the series. The matrix $\boldsymbol{r}_B$ and the vector $\boldsymbol{y}$ of coefficients to be determined are defined as

$$\boldsymbol{r}_B = \{\boldsymbol{r}_0, \boldsymbol{r}_1, \boldsymbol{r}_2\} \qquad \boldsymbol{y}^T = \{y_0, y_1, y_2\} \tag{11}$$

Assuming the binomial series (a similar approach has been demonstrated recently for the Taylor series),[7] the terms for $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are given by

$$\boldsymbol{r}_1 = -\boldsymbol{B}\boldsymbol{r}_0, \qquad \boldsymbol{r}_2 = -\boldsymbol{B}\boldsymbol{r}_1 \tag{12}$$

where matrix $\boldsymbol{B}$ is defined as

$$\boldsymbol{B} \equiv \boldsymbol{K}_0^{-1}\Delta\boldsymbol{K} \tag{13}$$

For purposes of simplicity, the presentation of Eq. (13) is only symbolic. In general, the inverse $\boldsymbol{K}_0^{-1}$ is not calculated, as is shown later in this section.

Given $\boldsymbol{K}_0$ and $\boldsymbol{r}_0$, the following procedure is carried out to evaluate the displacements and the stresses for any change $\Delta\boldsymbol{K}$ in the stiffness matrix.[28]

1) The modified matrix $\boldsymbol{K} = \boldsymbol{K}_0 + \Delta\boldsymbol{K}$ and the basis vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ [Eq. (12)] are introduced. In cases in which the load vector is modified because of changes in the design, this modification can be considered in calculation of the basis vectors.

2) The reduced (3×3) matrix $\boldsymbol{K}_R$ and the reduced (3×1) vector $\boldsymbol{R}_R$ are calculated by the expressions

$$\boldsymbol{K}_R = \boldsymbol{r}_B^T\boldsymbol{K}\boldsymbol{r}_B \qquad \boldsymbol{R}_R = \boldsymbol{r}_B^T\boldsymbol{R} \tag{14}$$

3) The coefficients $\boldsymbol{y}$ are calculated by solving the set of (3×3) equations

$$\boldsymbol{K}_R\boldsymbol{y} = \boldsymbol{R}_R \tag{15}$$

4) The final displacements and stresses are evaluated by Eqs. (10) and (6), respectively.

In the procedure described above, second-order series approximations have been assumed; therefore, it is called CA of order 2 (CA2). If first-order approximations are assumed (only two basis vectors), the coefficients $\boldsymbol{y}$ in step 3 are calculated by solving a set of (2×2) equations and the procedure is called, accordingly, CA of order 1 (CA1). The effectiveness of this approach has been demonstrated in several studies.[3–8] It has been found that high-quality approximations can be achieved for large changes in the design.

Exact Solution

For cross-sectional changes in truss structures, the exact solution is achieved if for each changed member a corresponding basis vector is assumed.[9] Specifically, for simultaneous changes in m members, the exact solution is given by

$$\boldsymbol{r} = \boldsymbol{r}_0 + \sum_{i=1}^{m} y_i\boldsymbol{r}_i \tag{16}$$

in which the basis vectors are defined as

$$\begin{aligned} \boldsymbol{r}_0 &= \text{initial displacements} \\ \boldsymbol{r}_i &= \boldsymbol{K}_0^{-1}\Delta\boldsymbol{K}_i\boldsymbol{r}_0 \qquad i = 1, \ldots, m \end{aligned} \tag{17}$$

and $\Delta\boldsymbol{K}_i$ is the contribution of the ith member to $\Delta\boldsymbol{K}$. This procedure is efficient in cases in which a change has been made in a limited number of members. If some of the basis vectors are linearly dependent, the exact solution is achieved for a smaller number of basis vectors. Exact solutions are also achieved in other cases such as scaling of the design.[18]

Number of DOF Is Increased

Establishing a Modified Initial Design (MID)

Adding a joint to the structure increases the number of DOF. Therefore, it is necessary first to expand the basis vectors and to introduce an MID, so that the new DOF are included in the analysis model. The MID can be selected so that reanalysis will be convenient and not necessarily a particular modified design. Any requested design can be analyzed at a later stage by corresponding changes. Once the MID is introduced, it is then possible to analyze conveniently modified structures that result from addition or deletion of members, keeping the number of DOF unchanged.

The MID and the expanded basis vectors can be established in several ways. A simple and convenient procedure for introducing and reanalyzing the MID is demonstrated in this section. By this approach, the MID is formed by adding a new joint and horizontal and vertical members connecting this joint with existing joints in the structure. When these members are not needed, they can be eliminated from the structure later while applying changes to the MID. In addition, a convenient temporary location for the new joint can be selected. Then it is possible to modify the joint coordinates by the procedure described later.

Considering a given initial design and adding a new joint and horizontal and vertical members, the equilibrium equations of the resulting MID are given by Eq. (7). Considering the modified stiffness matrix of Eq. (9), its inverse is a flexibility matrix $\boldsymbol{F}_M$, consisting of the corresponding submatrices

$$\boldsymbol{K}_M^{-1} = \boldsymbol{F}_M = \begin{bmatrix} \boldsymbol{F}_{00} & \boldsymbol{F}_{0M} \\ \boldsymbol{F}_{M0} & \boldsymbol{F}_{MM} \end{bmatrix} \tag{18}$$

Because only a horizontal member and a vertical member are connected to the new joint, the various submatrices are readily available. Specifically, it can be observed that

$$\begin{aligned} \boldsymbol{F}_{00} &= \boldsymbol{K}_0^{-1} \\ \boldsymbol{F}_{0M} = \boldsymbol{F}_{M0}^T &= \text{corresponding rows and columns of } \boldsymbol{K}_0^{-1} \\ \boldsymbol{F}_{MM} &= (\text{flexibility matrix of the new members, } \boldsymbol{K}_{MM}^{-1}) \\ &\quad + (\text{corresponding elements of } \boldsymbol{K}_0^{-1}) \end{aligned} \tag{19}$$

In general, it is not necessary to calculate the inverse, as shown later in this section. However, in cases in which $\boldsymbol{K}_0^{-1}$ is given from the initial analysis, all submatrices in Eq. (18) can be determined in a simple way.

The modified initial displacement vector is given by

$$\boldsymbol{r}_M = \begin{Bmatrix} \boldsymbol{r}_0 \\ \Delta\boldsymbol{r}_0 \end{Bmatrix} \tag{20}$$

in which $\Delta\boldsymbol{r}_0$ is the subvector of displacements of the new joint. Because there is no force in the members connected to the added joint, the elements of $\Delta\boldsymbol{r}_0$ are simply the corresponding displacements of the existing joints, as demonstrated later by numerical examples.

The procedure presented for addition of a single joint can readily be extended to the general case of addition of several joints. In the latter case, the initial location of all added joints may be assumed to be identical. The geometry of the structure is then modified as necessary.

Further Changes

Once the MID has been introduced, evaluation of the displacements for further changes in the topology and the geometry of the structure is straightforward. Considering the given initial value of the inverse $\boldsymbol{K}_M^{-1}$ [Eq. (18)] and the initial displacements $\boldsymbol{r}_M$ [Eq. (20)], the modified basis vectors are determined for any assumed $\Delta\boldsymbol{K}$ by

$$\boldsymbol{r}_{1M} = -\boldsymbol{K}_M^{-1}\Delta\boldsymbol{K}\boldsymbol{r}_M \qquad \boldsymbol{r}_{2M} = -\boldsymbol{K}_M^{-1}\Delta\boldsymbol{K}\boldsymbol{r}_{1M} \tag{21}$$

In this section, the solution procedure for various types of layout modifications, where the number of DOF is not increased, is briefly described. Members can be deleted from the structure by assuming zero cross sections. Note that a joint is automatically eliminated if all members connected to the joint are deleted from the structure. Addition of members is considered by assuming certain cross sections for such members. As noted earlier, the exact solution can be achieved if the basis vectors are determined by Eq. (17). Because the number of basis vectors is $m + 1$, this procedure is efficient only in the case of changes in a limited number of members.

The exact solution can also be efficiently achieved for geometrical modifications in the case of changes in a small number of members. The exact solution can be obtained by viewing these changes as corresponding topological modifications. Modifying, for example, the coordinates of a single joint, it is possible to obtain the exact solution for the new design by viewing the change in the geometry as the following two successive changes in the topology:

1) All members connected to that joint are eliminated.
2) New members are added at the modified location.

This procedure can also be used when it is necessary to modify the coordinates of a new joint added to the initial design to form the MID, as demonstrated in the numerical examples that follow.

Computational Considerations

In this section, some computational considerations related to the solution procedure are briefly discussed. Consider first the two possible methods of calculating the basis vectors: the Taylor series and the binomial series. Assuming the common first-order Taylor series expansion, each redesign involves very few operations. This is probably the most efficient reanalysis model. Second-order Taylor series expansion is often not practical because of the large computational effort involved in calculation of the second-order derivative matrices. An exception is the common case of homogeneous displacement functions, where the Taylor series and the binomial series become equivalent.

The advantage of using the binomial series is that, unlike the Taylor series, calculation of derivatives is not required. This makes the method more attractive in general applications where derivatives are not available. Assuming the binomial series, it has been shown in previous studies[3] that calculation of the basis vectors involves only forward and backward substitutions if $\boldsymbol{K}_0$ is given in the decomposed form of Eq. (2) from the initial analysis. Thus, the second-order terms can readily be calculated.

As to the selection of the number of basis vectors to be considered, it has been shown previously[3–8] that, in general, three basis vectors (second-order approximations) provide better results than two basis vectors (first-order approximations). This makes the binomial series more favorable in cases where high-quality approximations are important.

It is instructive to note that calculation of inverse matrices during the solution process is not necessary. Assuming that $\boldsymbol{K}_0$ is given in the decomposed form of Eq. (2) from the initial analysis, then calculation of $\boldsymbol{r}_0$ involves only forward and backward substitutions. As noted above, calculation of the remaining basis vectors ($\boldsymbol{r}_1$ and $\boldsymbol{r}_2$) involves similar algebraic operations. Finally, factorization of the expanded stiffness matrix [Eq. (9)], given that of the initial matrix [Eq. (2)], also requires very few operations.

Numerical Examples

Establishing an MID

To illustrate the solution procedure for the case of addition of members and joints, consider the initial seven-bar truss shown in Fig. 1a. The truss is subjected to a single loading condition of two concentrated loads, the modulus of elasticity is $E = 30{,}000$, and the six analysis unknowns are the horizontal (to the right) and the vertical (downward) displacements at joints 1, 2, and 3, respectively.

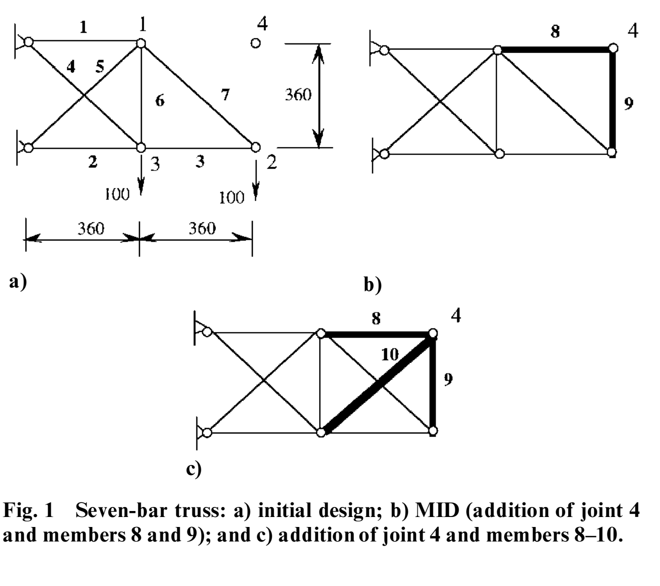

Fig. 1 Seven-bar truss: a) initial design; b) MID (addition of joint 4 and members 8 and 9); and c) addition of joint 4 and members 8–10.

Assuming initial cross-sectional areas $\boldsymbol{X}_0 = \mathbf{1.0}$, then the initial inverse of the stiffness matrix, the initial load vector, and the initial displacement vector are given by

$$\boldsymbol{K}_0^{-1} = \frac{1}{83.333}\begin{bmatrix} 0.89 & & & & & \\ 0.56 & 2.14 & & \text{Symmetric} & & \\ -0.12 & -0.44 & 1.88 & & & \\ 1.56 & 3.14 & -2.44 & 9.97 & & \\ -0.12 & -0.44 & 0.88 & -1.44 & 0.88 & \\ 0.44 & 1.69 & -0.56 & 2.69 & -0.56 & 2.14 \end{bmatrix} \tag{22}$$

$$\boldsymbol{R}^T = [0, 0, 0, 100, 0, 100] \tag{23}$$

$$\boldsymbol{r}_0^T = \{2.40, 5.80, -3.60, 15.19, -2.40, 5.80\} \tag{24}$$

Adding the new joint 4 and the horizontal and vertical members 8 and 9 (connecting this joint with the existing joints 1 and 2, respectively), the resulting MID is shown in Fig. 1b. To introduce the inverse of the modified stiffness matrix [Eqs. (18) and (19)], we note that the submatrix $\boldsymbol{F}_{00} = \boldsymbol{K}_0^{-1}$ is already given from analysis of the initial design [Eq. (22)]. In addition, the elements of $\boldsymbol{F}_{M0}$ are simply rows one and four of $\boldsymbol{K}_0^{-1}$. (Similarly, the elements of $\boldsymbol{F}_{0M}$ are columns one and four of $\boldsymbol{K}_0^{-1}$.) That is,

$$\begin{aligned}\boldsymbol{F}_{M0} &= \boldsymbol{F}_{0M}^T \\ &= \frac{1}{83.333}\begin{bmatrix} 0.89 & 0.56 & -0.12 & 1.56 & -0.12 & 0.44 \\ 1.56 & 3.14 & -2.44 & 9.97 & -1.44 & 2.69 \end{bmatrix}\end{aligned} \tag{25}$$

The elements of $\boldsymbol{F}_{MM}$ can readily be determined by adding the inverse of the stiffness of the new joint (a diagonal 2×2 matrix) to the corresponding elements of $\boldsymbol{K}_0^{-1}$, namely, the elements 11, 14, 41, 44 in Eq. (22). The resulting submatrix is

$$\begin{aligned}\boldsymbol{F}_{MM} &= \frac{1}{83.333}\left(\begin{bmatrix} 0.89 & 1.56 \\ 1.56 & 9.97 \end{bmatrix} + \begin{bmatrix} 1.0 & \\ & 1.0 \end{bmatrix}\right) \\ &= \frac{1}{83.333}\begin{bmatrix} 1.89 & 1.56 \\ 1.56 & 10.97 \end{bmatrix}\end{aligned} \tag{26}$$

Finally, the modified initial displacement vector is

$$\begin{aligned}\boldsymbol{r}_M^T &= \{\boldsymbol{r}_0^T, \Delta\boldsymbol{r}_0^T\} \\ &= \{2.40, 5.80, -3.60, 15.19, -2.40, 5.80, 2.40, 15.19\}\end{aligned} \tag{27}$$

[Eq. (20)], in which the displacements of the new joint, $\Delta\boldsymbol{r}_0$, are the corresponding given displacements

$$r_{M7} = r_1 \qquad r_{M8} = r_4 \tag{28}$$

Further Topological Modifications

Considering the MID of Fig. 1b and adding member 10, we obtain the 10-bar truss shown in Fig. 1c. Because only a single member was added, the exact solution can be determined by considering only the basis vectors $\boldsymbol{r}_M$, $\boldsymbol{r}_{1M}$. The vector $\boldsymbol{r}_M$ is given by Eq. (27), whereas the vector $\boldsymbol{r}_{1M}$ is calculated by Eq. (21):

$$\boldsymbol{r}_{1M}^T = \{0.19, 0.72, -1.44, 6.94, 0.19, -0.72, -1.44, 8.56\} \tag{29}$$

The coefficients $\boldsymbol{y}$, calculated by Eq. (15), are $y_0 = 1.0$ and $y_1 = -0.2964$ and the resulting exact displacements are

$$\boldsymbol{r}^T = \{2.34, 5.58, -3.17, 13.13, -2.46, 6.01, 2.83, 12.65\} \tag{30}$$

[Eq. (16)]. Other members can be deleted from the MID or added to it and the exact solution is achieved in a similar way. If the number of modified elements is large, then an approximate solution can efficiently be achieved, as is demonstrated in the next section.

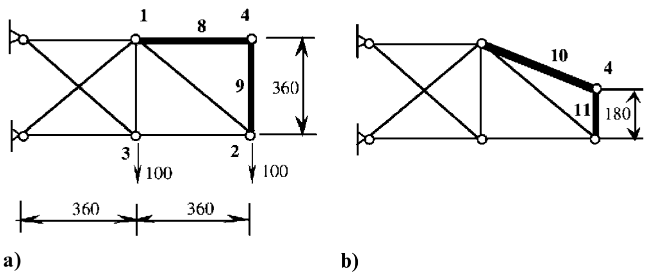

Fig. 2 Nine-bar truss: a) initial design and b) change in the geometry, viewed as elimination of members 8 and 9, and addition of members 10 and 11.

Further Geometrical Modifications

Consider again the MID of Fig. 1b, shown also in Fig. 2a. To calculate displacements for the modified geometry shown in Fig. 2b, members 8 and 9 connected to joint 4 are deleted from the structure and new members 10 and 11 are added in the new location of the joint. The matrix of changes in the stiffness is

$$\Delta \boldsymbol{K} = \Delta \boldsymbol{K}(8, 9) + \Delta \boldsymbol{K}(10, 11) = \begin{bmatrix} \Delta \boldsymbol{K}_{11} & \Delta \boldsymbol{K}_{12} \\ \Delta \boldsymbol{K}_{21} & \Delta \boldsymbol{K}_{22} \end{bmatrix} \tag{31}$$

in which

$$\Delta \boldsymbol{K}_{11} = 83.333 \begin{bmatrix} -0.2844 & 0.3578 & 0 & 0 \\ 0.3578 & 0.1789 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Delta \boldsymbol{K}_{22} = 83.333 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.2844 & 0.3578 \\ 0 & 0 & 0.3578 & 1.1789 \end{bmatrix} \tag{32}$$

$$\Delta \boldsymbol{K}_{21} = \Delta \boldsymbol{K}_{12}^T = 83.333 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.2844 & -0.3578 & 0 & 0 \\ -0.3578 & -0.1789 & 0 & -1 \end{bmatrix}$$

Because four members have been changed, four corresponding basis vectors are needed to achieve the exact solution. However, it has been found that in various cases a smaller number of vectors is sufficient. In the present case, the matrices corresponding to the two vertical members 9 and 11 are linearly dependent; therefore, only three basis vectors are required to achieve the exact displacements

$$\boldsymbol{r}^T = \{2.40, 5.80, -3.60, 15.19, -2.40, 5.80, -2.30, 15.19\} \tag{33}$$

General Layout Modifications

Solution Procedure

In previous papers, approximate solutions[6] and exact solutions[9] have been demonstrated for various cases of elimination and addition of truss members without increasing the number of DOF. In this study, reanalysis models for several cases that have not previously been solved are presented. A simple reanalysis procedure has been developed for cases in which the number of DOF is increased. It has been shown that exact solutions can be achieved for all types of layout modifications in trusses, including addition of members and joints and modifications in the geometry of the structure. Because each changed member requires a corresponding basis vector, the exact solution procedure is efficient only in cases of a change in a limited number of members. In addition, exact solutions usually can be achieved only for trusses.[28] For changes in numerous members, an approximate solution procedure can be used.

An effective reanalysis procedure, suitable for all types of layout modifications, is described here. The method developed is used for general layout modifications, including changes in the number of members and joints and in the joint coordinates. Evaluation of the modified displacements and stresses involves the following calculations:

1) Establishing an MID. This step is carried out only when a joint is added to the structure. A simple procedure for introducing the MID is to add horizontal and vertical members connecting this joint with existing joints in the structure, but other possibilities might be considered. It has been noted that the MID can be selected so that reanalysis will be convenient and not necessarily a particular modified design. Any requested design can be analyzed at a later stage by corresponding changes. Assuming horizontal and vertical members connected to the new joint, the modified load vector $\boldsymbol{R}_M$ is given by Eq. (8) and the inverse of the modified stiffness matrix $\boldsymbol{K}_M^{-1}$ is determined by Eq. (18). The various submatrices of $\boldsymbol{K}_M^{-1}$ and the modified initial displacement vector are readily available [Eqs. (19) and (20), respectively]. As noted earlier, calculation of the inverse $\boldsymbol{K}_M^{-1}$ usually is not needed. In cases in which the added members are not needed, they can be eliminated by the procedure described in steps 2 and 3 as follows.

2) Introducing the basis vectors. Given the modified decomposed stiffness matrix and the displacement vector for the MID, the matrix of changes in the stiffness $\Delta \boldsymbol{K}$ is determined for any assumed change in the structure. It should be emphasized that $\Delta \boldsymbol{K}$ might represent changes of a different nature, including various combinations of topological, geometrical, and cross-sectional modifications. The exact solution can be achieved in the case of changes in a limited number of truss members. For cross-sectional changes, the exact solution is achieved if for each changed member a corresponding basis vector is assumed. A similar procedure also can be used in cases of addition and deletion of members as well as changes in joint coordinates, all of which can be viewed as modifications in cross sections. For changes in numerous members, an approximate solution can be efficiently obtained by considering only two or three basis vectors (CA1 and CA2, respectively). The basis vectors are calculated by Eq. (21).

3) Evaluation of the modified displacements and stresses. This step is carried out by the procedure described earlier. The reduced stiffness matrix $\boldsymbol{K}_R$ and the reduced load vector $\boldsymbol{R}_R$ are first calculated by Eqs. (14). The coefficients $\boldsymbol{y}$ are then calculated by solving the reduced set of Eq. (15), and the resulting displacements and stresses are finally evaluated by Eqs. (10) and (6), respectively.

Numerical Examples

Ten-Bar Truss

To illustrate results for the case of simultaneous elimination of members and joints and modifications in geometry, consider the initial 10-bar truss design shown in Fig. 3a. The truss is subjected to a single loading condition of two concentrated loads. The initial cross-sectional areas are $\boldsymbol{X}_0 = \mathbf{1.0}$, the modulus of elasticity is

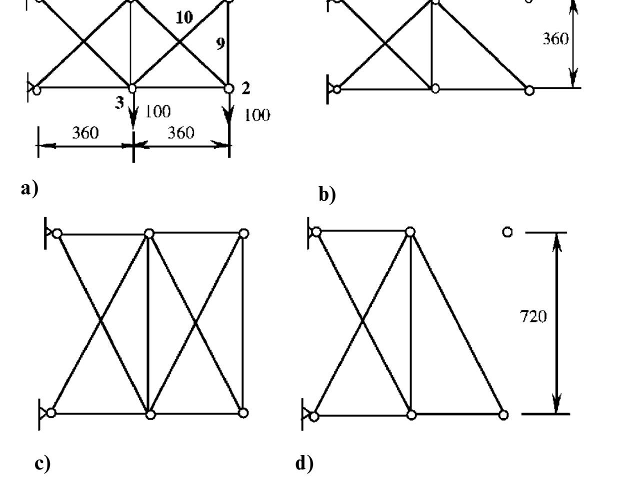

Fig. 3 Ten-bar truss: a) initial design; b) elimination of members 8–10; c) increasing the depth by 100%; and d) elimination of members 8–10 and increasing the depth.

Table 1 Results, eliminating members 8–10

Method	Displacements							
Exact	2.40	5.79	−3.60	15.18	−2.40	5.79	*	*
CA2	2.40	5.79	−3.60	15.18	−2.40	5.79	*	*
CA1	2.32	5.47	−3.67	14.83	−2.48	6.11	*	*

*Joint 4 is eliminated.

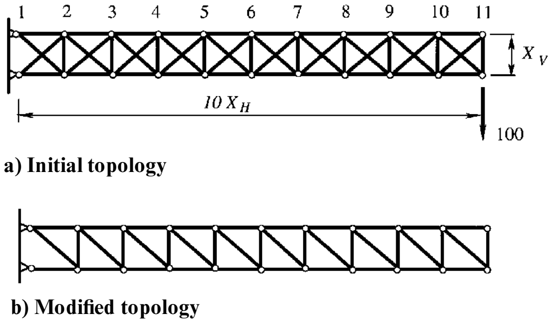

Fig. 4 Fifty-bar truss.

$E = 30{,}000$, and the eight analysis unknowns are the horizontal (to the right) and the vertical (downward) displacements at joints 1, 2, 3, and 4, respectively. Results for three different cases of modified designs due to changes in the structural layout are demonstrated:

1) Elimination of members 8–10 (Fig. 3b).

2) Changing the geometry by increasing the depth of the truss by 100% (Fig. 3c). This is a change in 6 members but, to achieve the exact solution, it can be viewed as a change in 12 members (elimination of 6 members and addition of 6 members).

3) Combination of the previous two cases, that is, simultaneous elimination of members 8–10 and increasing the depth by 100% (Fig. 3d).

Approximate results achieved by CA1 and CA2 (considering only two and three basis vectors, respectively) are summarized in Tables 1, 2, and 3. The results achieved by the CA2 are very close to the exact solution. Good approximations have also been achieved by the CA1.

Fifty-Bar Truss

To illustrate first-order approximations for a structure with a larger number of DOF, consider the cantilever truss shown in Fig. 4a and subjected to a single load at the end. The initial cross sections are $\boldsymbol{X}_0 = \mathbf{1.0}$, the modulus of elasticity is $E = 10{,}000$, and the 40 unknowns are the X direction (to the right) and the Y direction (downward) displacements in the free joints. Considering two geometric variables, X_V and X_H, and assuming the initial geometry $X_V = X_H = 1.0$, three cases of layout modifications have been solved.

1) Topological modifications, where 10 diagonal members have been eliminated to obtain the topology shown in Fig. 4b.

2) Geometrical modifications, where the modified geometry is given by $X_V = 1.2$ (a change of 20% in the depth).

3) Geometrical modifications, where the modified geometry is given by $X_V = 2.0$, $X_H = 1.9$ (a change of 100% in the depth and 90% in the width).

In cases 2 and 3, the stiffness coefficients of 30 members have been changed; therefore, exact reanalysis is not efficient. Assuming the CA1, the displacements are evaluated by

$$\boldsymbol{r} = y_0\,\boldsymbol{r}_0 + y_1\,\boldsymbol{r}_1 = y_0\,\boldsymbol{r}_0 - y_1\,\boldsymbol{B}\boldsymbol{r}_0$$

and the results are given in Table 4. The approximate results achieved for the significant topological changes in case 1 are very close to the exact solution. Comparing the results obtained for the two cases of geometrical modifications (cases 2 and 3) shows that better approximations have been achieved in case 3 for larger changes in the geometry. The better results in case 3 are attributed to the fact that the modified geometry is relatively close to a scaled geometry ($X_V = X_H$), for which the CA1 provides the exact solution.

Table 2 Results, increasing the depth

Method	Displacements							
Exact	1.15	3.67	−1.66	7.36	−1.25	4.24	1.34	6.60
CA2	1.14	3.67	−1.68	7.35	−1.24	4.25	1.34	6.62
CA1	1.17	3.78	−1.61	7.29	−1.28	4.27	1.26	6.72

Table 3 Results, eliminating members 8–10 and increasing the depth

Method	Displacements							
Exact	1.20	3.95	−1.80	8.80	−1.20	3.96	*	*
CA2	1.19	3.93	−1.81	8.79	−1.20	3.97	*	*
CA1	1.25	3.99	−1.89	8.59	−1.25	4.12	*	*

*Joint 4 is eliminated.

Table 4 Displacements, 50-bar truss, and various layout modifications

		Case 1		Case 2		Case 3	
Joint	Direction	Exact	CA1	Exact	CA1	Exact	CA1
2	X	0.090	0.089	0.079	0.089	0.200	0.204
	Y	0.138	0.143	0.079	0.106	0.240	0.247
3	X	0.170	0.169	0.150	0.160	0.379	0.384
	Y	0.457	0.460	0.283	0.347	0.878	0.897
4	X	0.240	0.239	0.213	0.216	0.537	0.541
	Y	0.935	0.938	0.599	0.693	1.871	1.901
5	X	0.300	0.210	0.267	0.260	0.674	0.677
	Y	1.554	1.555	1.011	1.117	3.175	3.213
6	X	0.350	0.350	0.312	0.293	0.789	0.790
	Y	2.292	2.293	1.507	1.602	4.744	4.786
7	X	0.390	0.390	0.350	0.317	0.884	0.883
	Y	3.131	3.131	2.072	2.130	6.535	6.576
8	X	0.420	0.420	0.379	0.335	0.958	0.954
	Y	4.049	4.048	2.693	2.689	8.503	8.538
9	X	0.440	0.440	0.400	0.346	1.010	1.005
	Y	5.028	5.026	3.356	3.268	10.604	10.629
10	X	0.450	0.450	0.413	0.353	1.042	1.036
	Y	6.046	6.044	4.046	3.861	12.794	12.805
11	X	0.450	0.450	0.416	0.354	1.052	1.045
	Y	7.075	7.074	4.748	4.454	15.024	15.019

Concluding Remarks

In this paper, reanalysis models for some cases that have not been solved in previous studies have been developed. In particular, approximate and exact solutions for the case of addition of members and joints, and exact solutions for modifications in the geometry of the structure, are presented. A general procedure for reanalysis of structures subjected to various layout modifications is then introduced. The procedure is suitable for all types of simultaneous layout changes, including the general case when some members and joints are added, other members and joints are deleted, and some joint coordinates are modified. The presentation is focused on the most challenging case of addition of joints, in which the structural model and the number of DOF are changed. To overcome the special difficulties involved in the solution process, a simple reanalysis of an intermediate MID is first carried out. Further changes are then applied to the latter design.

It has been shown that, for changes in a small number of members of a truss structure, an exact solution is readily achieved. For changes in numerous members of a general structure, an efficient approximate solution can be obtained. High-quality approximations have been demonstrated for various types of layout modifications.

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